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New exact analytical solutions for Stokes' first problem of Maxwell fluid with fractional derivative approach

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ABSTRACT

The unsteady flow of an incompressible Maxwell fluid with fractional derivative induced by a sudden moved plate has been studied using Fourier sine and Laplace transforms. The obtained solutions for the velocity field and shear stress, written in terms of generalized G functions, are presented as sum of the similar Newtonian solutions and the corresponding non-Newtonian contributions. The non-Newtonian contributions, as expected, tend to zero for $\lambda \to 0$. Furthermore, the solutions for ordinary Maxwell fluid, performing the same motion, are obtained as limiting cases of general solutions and verified by comparison with previously known results. Finally, the influence of the material and the fractional parameters on the fluid motion, as well as a comparison among fractional Maxwell, ordinary Maxwell and Newtonian fluids is also analyzed by graphical illustrations.

1. Introduction

Stokes' celebrated paper on pendulums in 1851, in which he described a kind of classical problems of the impulsive and the oscillatory motion of an infinite plate in its own plane, is familiar to almost every student of fluid mechanics. Nowadays Stokes' first problem (or Rayleigh-type flow) term is used for flows over a plane wall which is initially at rest and is suddenly set into motion in its own plane with a constant velocity. Stokes' second problem is used for harmonic vibration of the plane. At present, many researchers and applied scientists pay more and more attention to Stokes' problems due to their wide application in sciences, engineering and technology. The exact solutions are very important in every areas of fluid mechanics not only because they are solutions for some fundamental flows, but because they serve as accuracy to check the experimental, numerical and asymptotic methods. We mention here some recent attempts regarding the exact analytical solutions for Stokes' first problem of non-Newtonian fluids [1–10].

From the last decade, the fractional calculus has encountered much success in description of complex dynamics such as relaxation, oscillation, wave and viscoelastic behavior. Several authors suggested that the integral-order models for viscoelastic materials seem to be inadequate, especially from the qualitative point of view. In the same time they proposed fractional-order laws of deformation for modeling the viscoelastic behavior of real materials. One of them

$$\sigma(t) = \mu_s D_t^1[\varepsilon(t)] + \left[\frac{3}{2}(\mu_0 - \mu_s) nKT\right]^{\frac{1}{2}} D_t^{\frac{1}{2}}[\varepsilon(t)],$$

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due to Rouse [11], is used in the molecular theory for dilute polymer solutions. Here σ is the stress, ε the strain, μ_s is the steady-flow viscosity of the solvent, μ_0 is the steady-flow viscosity of the solution, n is the number of molecules, K is the Boltzmann constant, T is the absolute temperature and D_t^{α} is the Riemann–Liouville fractional differential operator [12,13]. Ferry et al. [14], modified the Rouse theory in concentrated polymer solutions and polymer solids with no cross-linking and obtained that

$$\sigma(t) = \left[\frac{3\mu\rho RT}{2M}\right]^{\frac{1}{2}} D_t^{\frac{1}{2}}[\varepsilon(t)],$$

where μ is the viscosity, ρ the density, R is the universal gas constant and M the molecular weight. However, the use of fractional derivatives within the context of viscoelasticity was first proposed by Germant [15]. Subsequently, the theory was extended and Bagley and Torvik [16] demonstrated that the theory of viscoelasticity of coiling polymers predicts constitutive relations with fractional derivatives. The interest in this subject also resulted from a practical problem, that of predicting the dynamic response of viscous dampers by only knowing the constitutive relationship of the fluid contained in these devices. Modeling the behavior of viscous dampers is an increasingly important problem because of their wide range of applicability. The fluid used in the damper by Makris [17] is a form of silicon gel whose mass density is slightly less than that of water. Attempts were made to fit the properties of the fluid with conventional models of viscoelasticity, but it was not possible to achieve satisfactory fit of the experimental data over the entire range of frequencies. A very good fit of the experimental data was achieved when the Maxwell model was used with its first-order derivatives replaced by fractional-order derivatives. The shear stress–strain relationship in the fractional derivative Maxwell model, proposed by Makris et al. [18], is

$$\tau + \lambda D^r(\tau) = \mu D^q(\gamma)$$

where τ and γ are shear stress and strain, λ and μ are material constants and D^r is a fractional derivative operator of order r with respect to time. This model is a special case of the more general model of Bagley and Torvik [19]. It collapses to the conventional Maxwell model with $r = q = 1$, in which case λ and μ became the relaxation time and dynamic viscosity, respectively.

Consequently, the fractional calculus approach to viscoelasticity for the study of viscoelastic material properties is justified, at least for polymer solutions and for polymer solids without cross-linking. In the meantime, a lot of exact solutions corresponding to different motions of non-Newtonian fluids with fractional derivatives have been established, but we mention here only the flow over/between planar like domains [20–29]. Furthermore, the one-dimensional fractional derivative Maxwell model has been found very useful in modeling the linear viscoelastic response of some polymers in the glass transition and the glass state [30]. In other cases it has been shown that the governing equations employing fractional derivatives are also linked to molecular theories [31]. Furthermore, it is worth pointing out that Palade et al. [32] developed a fully objective constitutive equation for an incompressible fluid reducible to the linear fractional derivative Maxwell model under small deformation hypothesis.

The aim of this communication is to find some new and simple results for Stokes' first problem of rate type fluids. More exactly, our interest is to find the velocity field and the shear stress corresponding to the motion of a Maxwell fluid due to a sudden moved plate. However, for completeness, we shall determine exact solutions for a larger class of such fluids. Consequently, motivated by the above remarks, we solve our problem for Maxwell fluids with fractional derivatives. The general solutions are obtained using the Fourier sine and Laplace transforms. They are presented in series form in term of the generalized $G_{a,b,c}(\bullet, t)$ functions, and presented as sum of the similar Newtonian solutions and the corresponding non-Newtonian contributions. The similar solutions for ordinary Maxwell fluids, can easily be obtained as limiting cases of general solutions and verified by comparison with the known results. Furthermore, the Newtonian solutions are also obtained as limiting cases of fractional and ordinary Maxwell fluids. Finally, the influence of the material and fractional parameters on the motion of fractional and ordinary Maxwell fluids is underlined by graphical illustrations. The difference among fractional Maxwell, ordinary Maxwell and Newtonian fluid models is also spotlighted. The Newtonian fluid is the swiftest and the fractional Maxwell fluid is the slowest.

2. Basic governing equations

The equations governing the flow of an incompressible fluid include the continuity equation and the momentum equation. In the absence of body forces, they are

$$\nabla \cdot \mathbf{V} = 0, \tag{1}$$

$$\nabla \cdot \mathbf{T} = \rho \frac{\partial \mathbf{V}}{\partial t} + \rho\big(\mathbf{V} \cdot \nabla\big)\mathbf{V}, \tag{2}$$

where ρ is the fluid density, $\mathbf{V}$ is the velocity field, t is the time and ∇ represents the gradient operator. The Cauchy stress $\mathbf{T}$ in an incompressible Maxwell fluid is given by [2,5,7]

$$\mathbf{T} = -p\mathbf{I} + \mathbf{S}, \qquad \mathbf{S} + \lambda(\dot{\mathbf{S}} - \mathbf{LS} - \mathbf{SL}^T) = \mu\mathbf{A}, \tag{3}$$

where $-p\mathbf{I}$ denotes the indeterminate spherical stress due to the constraint of incompressibility, $\mathbf{S}$ is the extra-stress tensor, $\mathbf{L}$ is the velocity gradient, $\mathbf{A} = \mathbf{L} + \mathbf{L}^T$ is the first Rivlin Ericksen tensor, μ is the dynamic viscosity of the fluid, λ is the relaxation time, the superscript T indicates the transpose operation and the superposed dot indicates the material time derivative. The model characterized by the constitutive equations (3) contains as special case the Newtonian fluid model for $\lambda \to 0$. The model (3) is consistent with some important microscopical models of polymers and its predictions of the normal-stress differences are qualitatively acceptable. It has been quite useful in the study of dilute polymeric fluids in viscoelasticity.

For the problem under consideration we assume a velocity field $\mathbf{V}$ and an extra-stress tensor $\mathbf{S}$ of the form

$$\mathbf{V} = \mathbf{V}(y,t) = u(y,t)\mathbf{i}, \qquad \mathbf{S} = \mathbf{S}(y,t), \tag{4}$$

where $\mathbf{i}$ is the unit vector along the x-coordinate direction. For these flows the constraint of incompressibility is automatically satisfied. If the fluid is at rest up to the moment $t = 0$, then

$$\mathbf{V}(y,0) = \mathbf{0}, \qquad \mathbf{S}(y,0) = \mathbf{0}, \tag{5}$$

and Eqs. (3) and (4) imply $S_{yy} = S_{yz} = S_{zz} = S_{xz} = 0$, and the meaningful equation

$$\left(1 + \lambda \frac{\partial}{\partial t}\right)\tau(y,t) = \mu \frac{\partial u(y,t)}{\partial y}, \tag{6}$$

where $\tau(y,t) = S_{xy}(y,t)$ is the non-zero shear stress. In the absence of body forces, the balance of linear momentum (2) reduces to

$$\frac{\partial \tau(y,t)}{\partial y} - \frac{\partial p}{\partial x} = \rho \frac{\partial u(y,t)}{\partial t}, \qquad \frac{\partial p}{\partial y} = 0, \qquad \frac{\partial p}{\partial z} = 0. \tag{7}$$

Eliminating τ between Eqs. (6) and $(7)_1$, we find the governing equation under the form

$$\left(1 + \lambda \frac{\partial}{\partial t}\right)\frac{\partial u(y,t)}{\partial t} = -\frac{1}{\rho}\left(1 + \lambda \frac{\partial}{\partial t}\right)\frac{\partial p}{\partial x} + \nu \frac{\partial^2 u(y,t)}{\partial y^2}, \tag{8}$$

where $\nu = \mu/\rho$ is the kinematic viscosity of the fluid.

The governing equations corresponding to an incompressible Maxwell fluid with fractional derivatives, performing the same motion in the absence of a pressure gradient in the flow direction, are (cf. [5,10,24,26,28])

$$(1 + \lambda^\alpha D_t^\alpha)\frac{\partial u(y,t)}{\partial t} = \nu \frac{\partial^2 u(y,t)}{\partial y^2}, \qquad (1 + \lambda^\alpha D_t^\alpha)\tau(y,t) = \mu \frac{\partial u(y,t)}{\partial y}, \tag{9}$$

where α is the fractional parameter, and the fractional differential operator so called Caputo fractional operator D_t^α defined by [12,13]

$$D_t^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{f'(\tau)}{(t-\tau)^\alpha} d\tau; \quad 0 \le \alpha < 1, \tag{10}$$

and $\Gamma(\bullet)$ is the Gamma function. In the following the system of fractional partial differential equations (9), with appropriate initial and boundary conditions, will be solved by means of Fourier sine and Laplace transforms. In order to avoid lengthy calculations of residues and contour integrals, the discrete inverse Laplace transform method will be used [20–29].

3. Statement of the problem

Consider an incompressible Maxwell fluid with fractional derivatives occupying the space lying over an infinitely extended plate which is situated in the (x,z) plane and perpendicular to the y-axis. Initially, the fluid is at rest and at the moment $t = 0^+$ the plate is impulsively brought to the constant velocity U in its plane. Due to the shear, the fluid above the plate is gradually moved. Its velocity is of the form $(4)_1$ while the governing equations are given by Eq. (9). The appropriate initial and boundary conditions are

$$u(y,0) = \frac{\partial u(y,0)}{\partial t} = 0; \qquad \tau(y,0) = 0, \quad y > 0, \tag{11}$$

$$u(0,t) = UH(t); \quad t \ge 0, \tag{12}$$

where $H(t)$ is the Heaviside function. Moreover, the natural conditions

$$u(y,t),\ \frac{\partial u(y,t)}{\partial y} \to 0 \quad \text{as } y \to \infty \text{ and } t > 0, \tag{13}$$

also have to be satisfied. They are consequences of the fact that the fluid is at rest at infinity and there is no shear in the free stream.

4. Solution of the problem

4.1. Calculation of the velocity field

In order to determine the exact analytical solution, we shall use the Fourier sine transform [33]. Multiplying both sides of Eq. $(9)_1$ by $\sqrt{2/\pi}\,\sin(y\xi)$, integrating the result with respect to y from 0 to infinity, and taking into account the boundary condition (12), we find that

$$\left(1+\lambda^\alpha D_t^\alpha\right)\frac{\partial u_s}{\partial t}+\nu\xi^2 u_s=U\nu\xi\sqrt{\frac{2}{\pi}}H(t);\quad \xi,t>0, \tag{14}$$

where the Fourier sine transform $u_s(\xi,t)$ of $u(y,t)$ defined by [33]

$$u_s(\xi,t)=\sqrt{\frac{2}{\pi}}\int_0^\infty u(y,t)\sin(y\xi)\,\mathrm{d}y, \tag{15}$$

has to satisfy the initial conditions

$$u_s(\xi,0)=\frac{\partial u_s(\xi,0)}{\partial t}=0;\quad \xi>0. \tag{16}$$

Applying the Laplace transform to Eq. (14), using the Laplace transform formula for sequential fractional derivatives [13] and taking into account the initial conditions (16), we find that

$$\bar{u}_s(\xi,q)=U\xi\sqrt{\frac{2}{\pi}}\frac{\nu}{q\left[q+\lambda^\alpha q^{\alpha+1}+\nu\xi^2\right]}. \tag{17}$$

In order to obtain $u_s(\xi,t)=\mathcal{L}^{-1}\{\bar{u}(\xi,q)\}$ and to avoid the lengthy and burdensome calculations of residues and contours integrals, we apply the discrete inverse Laplace transform method [20–29]. However, for a suitable representation of the velocity field, we first rewrite Eq. (17) in the equivalent form

$$\bar{u}_s(\xi,q)=U\sqrt{\frac{2}{\pi}}\frac{1}{\xi}\left[\frac{1}{q}-\frac{1}{q+\nu\xi^2}\right]-\nu U\xi\sqrt{\frac{2}{\pi}}\frac{q}{q+\nu\xi^2}F_1(\xi,q), \tag{18}$$

in which

$$F_1(\xi,q)=\frac{\lambda^\alpha q^{\alpha-1}}{q+\lambda^\alpha q^{\alpha+1}+\nu\xi^2}=\sum_{k=0}^\infty\left(\frac{-\nu\xi^2}{\lambda^\alpha}\right)^k\frac{q^{\alpha-k-2}}{(q^\alpha+\lambda^{-\alpha})^{k+1}}. \tag{19}$$

Inverting Eq. (18) by means of the Fourier sine formula [33], we find that

$$\bar{u}(y,q)=\frac{U}{q}-\frac{2U}{\pi}\int_0^\infty\frac{\sin(y\xi)}{\xi}\frac{\mathrm{d}\xi}{q+\nu\xi^2}-\frac{2\nu U}{\pi}\int_0^\infty\frac{q\xi\sin(y\xi)}{q+\nu\xi^2}F_1(\xi,q)\,\mathrm{d}\xi. \tag{20}$$

Introducing (19) into (20), inverting the result by means of the discrete inverse Laplace transform and using the convolution theorem for inverse Laplace transform and the known result [34, Eq. (97)]

$$\mathcal{L}^{-1}\left\{\frac{q^b}{(q^a-d)^c}\right\}=G_{a,b,c}(d,t);\qquad \mathrm{Re}\,(ac-b)>0,\qquad \mathrm{Re}(q)>0,\qquad \left|\frac{d}{q^a}\right|<1, \tag{21}$$

where the generalized $G_{a,b,c}(\bullet,t)$ function is defined by [34, Eqs. (101) and (99)]

$$G_{a,b,c}(d,t)=\sum_{j=0}^\infty\frac{d^j\Gamma(c+j)}{\Gamma(c)\Gamma(j+1)}\frac{t^{(c+j)a-b-1}}{\Gamma[(c+j)a-b]}, \tag{22}$$

we find the velocity field in the form

$$u(y,t)=u_N(y,t)-\frac{2\nu U}{\pi}\int_0^\infty\xi\sin(y\xi)f_1(\xi,t)\,\mathrm{d}\xi+\frac{2\nu^2 U}{\pi}\int_0^\infty\int_0^t\xi^3\sin(y\xi)f_1(\xi,s)\mathrm{e}^{-\nu\xi^2(t-s)}\,\mathrm{d}s\,\mathrm{d}\xi, \tag{23}$$

where

$$u_N(y,t)=U\left[1-\frac{2}{\pi}\int_0^\infty\frac{\sin(y\xi)}{\xi}\mathrm{e}^{-\nu\xi^2 t}\,\mathrm{d}\xi\right]=U\mathrm{erf}\,c\left(\frac{y}{2\sqrt{\nu t}}\right), \tag{24}$$

is the velocity field corresponding to a Newtonian fluid, and

$$f_1(\xi,t)=\sum_{k=0}^{\infty}\left(\frac{-\nu\xi^2}{\lambda^\alpha}\right)^k G_{\alpha,\alpha-k-2,k+1}\left(\frac{-1}{\lambda^\alpha},t\right), \tag{25}$$

is the inverse Laplace transform of $F_1(\xi,q)$. Of course, in view of the limit

$$\lim_{\lambda\to 0}\frac{1}{\lambda^\eta}G_{a,b,\eta}\left(\frac{-1}{\lambda},t\right)=\frac{t^{-b-1}}{\Gamma(-b)};\quad b<0, \tag{26}$$

it result that $f_1(\xi,t)\to 0$ for $\lambda\to 0$ and therefore $u(y,t)\to u_N(y,t)$.

4.2. Calculation of the shear stress

Applying the Laplace transform to Eq. (9)$_2$ and using the initial condition (11)$_3$, we find that

$$\overline{\tau}(y,q)=\frac{\mu}{1+\lambda^\alpha q^\alpha}\frac{\partial\overline{u}(y,q)}{\partial y}, \tag{27}$$

where $\overline{\tau}(y,q)$ is the Laplace transform of $\tau(y,t)$. In order to get $\partial\overline{u}(y,q)/\partial y$ we apply the inverse Fourier sine transform to Eq. (17) and find that

$$\overline{u}(y,q)=\frac{2U}{\pi}\int_0^\infty\frac{\nu\xi\sin(y\xi)}{q\left[q+\lambda^\alpha q^{\alpha+1}+\nu\xi^2\right]}\mathrm{d}\xi, \tag{28}$$

from which

$$\frac{\partial\overline{u}(y,q)}{\partial y}=\frac{2U}{\pi}\int_0^\infty\frac{\nu\xi^2\cos(y\xi)}{q\left[q+\lambda^\alpha q^{\alpha+1}+\nu\xi^2\right]}\mathrm{d}\xi. \tag{29}$$

Let us introduce (29) into (27), and use the decomposition

$$\frac{1}{q\left[q+\lambda^\alpha q^{\alpha+1}+\nu\xi^2\right]}=\frac{1}{\nu\xi^2}\left[\frac{1}{q}-\frac{1}{q+\nu\xi^2}\right]-H_1(\xi,q)H_2(\xi,q),$$

where

$$H_1(\xi,q)=\frac{q^{\alpha-1}}{\lambda^\alpha(q^\alpha+\lambda^{-\alpha})}\frac{1}{q+\nu\xi^2},\qquad H_2(\xi,q)=\frac{2\lambda^\alpha q+\lambda^{2\alpha}q^{\alpha+1}+\lambda^\alpha\nu\xi^2}{q+\lambda^\alpha q^{\alpha+1}+\nu\xi^2},$$

in order to obtain $\overline{\tau}(y,q)$ under the suitable form

$$\overline{\tau}(y,q)=\frac{2\mu U}{\pi}\int_0^\infty\cos(y\xi)\left[\frac{1}{q}-\frac{1}{q+\nu\xi^2}\right]\mathrm{d}\xi-\frac{2\nu\mu U}{\pi}\int_0^\infty\xi^2\cos(y\xi)H_1(\xi,q)H_2(\xi,q)\mathrm{d}\xi. \tag{30}$$

Inverting Eq. (30) by means of the discrete inverse Laplace transform and again using convolution theorem, we find the shear stress $\tau(y,t)$ under simple form

$$\tau(y,t)=\tau_N(y,t)-\frac{2\nu\mu U}{\pi}\int_0^\infty\int_0^t\xi^2\cos(y\xi)h_1(\xi,t-s)h_2(\xi,s)\mathrm{d}s\,\mathrm{d}\xi, \tag{31}$$

where

$$\tau_N(y,t)=-\frac{2\mu U}{\pi}\int_0^\infty\cos(y\xi)\mathrm{e}^{-\nu\xi^2t}\,\mathrm{d}\xi=-\frac{\mu U}{\sqrt{\pi\nu t}}\exp\left(-\frac{y^2}{4\nu t}\right), \tag{32}$$

is the shear stress corresponding to a Newtonian fluid, and

$$h_1(\xi,t)=\frac{1}{\lambda^\alpha}\int_0^t G_{\alpha,\alpha-1,1}\left(\frac{-1}{\lambda^\alpha},t\right)\mathrm{e}^{-\nu\xi^2(t-r)}\,\mathrm{d}r, \tag{33}$$

$$h_2(\xi,t)=\sum_{k=0}^{\infty}\left(\frac{-\nu\xi^2}{\lambda^\alpha}\right)^k\left\{2G_{\alpha,-k,k+1}\left(\frac{-1}{\lambda^\alpha},t\right)+\lambda^\alpha G_{\alpha,\alpha-k,k+1}\left(\frac{-1}{\lambda^\alpha},t\right)+\nu\xi^2G_{\alpha,-k-1,k+1}\left(\frac{-1}{\lambda^\alpha},t\right)\right\}, \tag{34}$$

are the inverse Laplace transforms of $H_1(\xi,q)$ and $H_2(\xi,q)$. Using the same limit (26) as before, we can easily prove that $\tau(y,t)\to\tau_N(y,t)$ for $\lambda\to 0$.

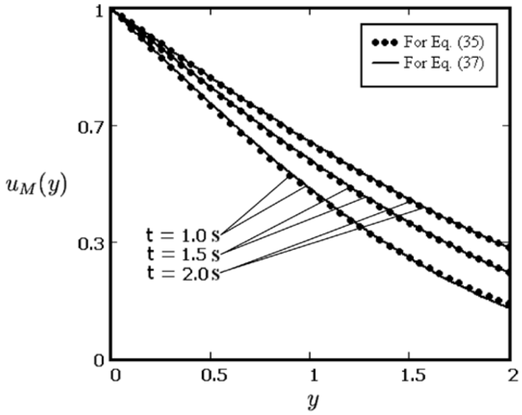

Fig. 1. Profiles of the velocity $u_M(y,t)$ given by Eqs. (35) and (37) for $U=1, h=1, \nu=1, \mu=1, \lambda=0.1, \alpha=1$ and different values of t.

4.3. The special case $\alpha \to 1$ (ordinary Maxwell fluid)

Making $\alpha \to 1$ in Eqs. (23) and (31), we obtain the velocity field

$$u_M(y,t) = u_N(y,t) - \frac{2\nu U}{\pi}\int_0^\infty \xi \sin(y\xi) f_{1M}(\xi,t)\,\mathrm{d}\xi + \frac{2\nu^2 U}{\pi}\int_0^\infty\int_0^t \xi^3 \sin(y\xi) f_{1M}(\xi,s)\mathrm{e}^{-\nu\xi^2(t-s)}\,\mathrm{d}s\,\mathrm{d}\xi, \tag{35}$$

and the associated shear stress

$$\tau_M(y,t) = \tau_N(y,t) - \frac{2\nu\mu U}{\pi}\int_0^\infty\int_0^t \xi^2 \cos(y\xi) h_{1M}(\xi,t-s) h_{2M}(\xi,s)\mathrm{d}s\,\mathrm{d}\xi, \tag{36}$$

corresponding to an ordinary Maxwell fluid performing the same motion. In the above relations

$$f_{1M}(\xi,q) = \sum_{k=0}^{\infty}\left(\frac{-\nu\xi^2}{\lambda}\right)^k G_{1,-k-1,k+1}\left(\frac{-1}{\lambda},t\right), \qquad h_{1M}(\xi,t) = \frac{\mathrm{e}^{-t/\lambda} - \mathrm{e}^{-\nu\xi^2 t}}{\lambda\nu\xi^2 - 1},$$

$$h_{2M}(\xi,t) = \sum_{k=0}^{\infty}\left(\frac{-\nu\xi^2}{\lambda}\right)^k \left\{2G_{1,-k,k+1}\left(\frac{-1}{\lambda},t\right) + \lambda G_{1,1-k,k+1}\left(\frac{-1}{\lambda},t\right) + \nu\xi^2 G_{1,-k-1,k+1}\left(\frac{-1}{\lambda},t\right)\right\},$$

and $u_M(y,t) \to u_N(y,t)$, $\tau_M(y,t) \to \tau_N(y,t)$ for $\lambda \to 0$. As a control of our solutions, we show by Fig. 1 that the diagrams of $u_M(y,t)$ given by Eq. (35) are almost identical to those corresponding to the solution

$$u_M(y,t) = U - \frac{2U}{\pi}\int_0^\infty \frac{\sin(y\xi)}{\xi}\left[\frac{q_2\mathrm{e}^{q_1 t} - q_1\mathrm{e}^{q_2 t}}{q_2 - q_1}\right]\mathrm{d}\xi, \quad q_1, q_2 = \frac{-1 \pm \sqrt{1-4\nu\lambda\xi^2}}{2\lambda} \tag{37}$$

obtained in [35, Eq. (11)] (see also [36, Eq. (15) for $\lambda_r = 0$]) by a different technique. Furthermore, as it results from Fig. 2, the diagrams of our solution (35) also are almost identical to those corresponding to Böhme's solution [37]

$$u(y,t) = \begin{cases} 0, & y > t\sqrt{\nu/\lambda} \\ U\exp\left(-\dfrac{y}{2\sqrt{\nu\lambda}}\right) + \dfrac{y}{2\sqrt{\nu\lambda}}\displaystyle\int_{y/\sqrt{\nu\lambda}}^{t/\lambda} \mathrm{e}^{-\xi/2}\frac{I_1\left(\frac{1}{2}\sqrt{\xi^2 - y^2/(\nu\lambda)}\right)}{\sqrt{\xi^2 - y^2/(\nu\lambda)}}\,\mathrm{d}\xi, & y < t\sqrt{\nu/\lambda}, \end{cases} \tag{38}$$

where I_1 is the modified Bessel function of first kind.

5. Numerical results and discussion

In the previous sections, we have established exact analytical solutions for Stokes' first problem for Maxwell fluids with fractional derivative. In order to capture some relevant physical aspects of the obtained results, several graphs are depicted in this section. Attention has been focused on analyzing the difference between the velocity as well as shear stress profiles of fractional and ordinary Maxwell fluids for the flow induced by a sudden moved plate. We interpret these results with respect to the variations of emerging parameters of interest.

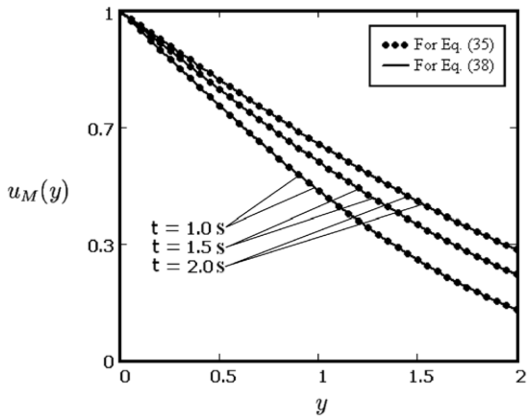

Fig. 2. Profiles of the velocity $u_M(y, t)$ given by Eqs. (35) and (38) for $U = 1, h = 1, \nu = 1, \mu = 1, \lambda = 0.1, \alpha = 1$ and different values of t.

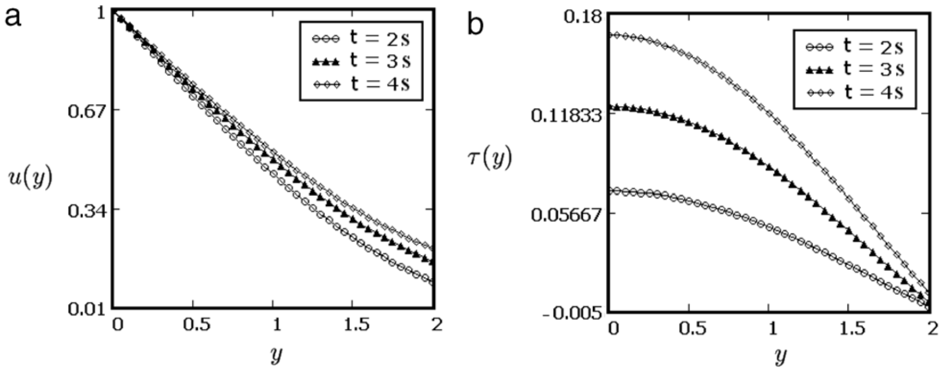

Fig. 3. Profiles of the velocity field $u(y, t)$ and the shear stress $\tau(y, t)$ for fractional Maxwell fluid given by Eqs. (23) and (31), for $\lambda = 2, \alpha = 0.4$ and different values of t.

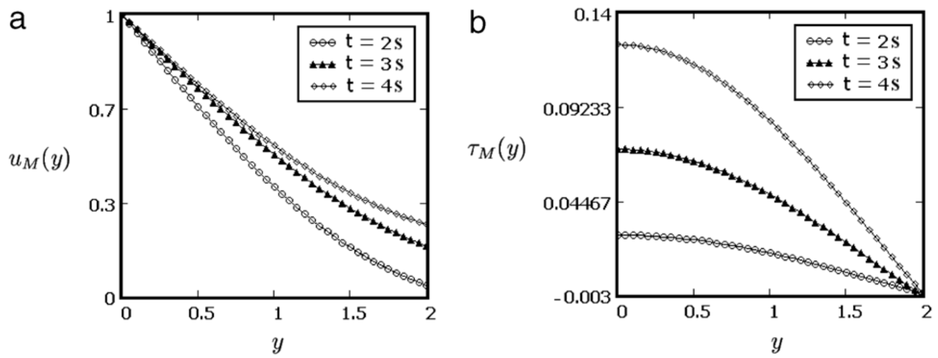

Fig. 4. Profiles of the velocity field $u_M(y, t)$ and the shear stress $\tau_M(y, t)$ for ordinary Maxwell fluid given by Eqs. (35) and (36), for $\lambda = 2, \alpha = 1$ and different values of t.

The diagrams of the velocity field $u(y, t)$ and $u_M(y, t)$, and the shear stresses $\tau(y, t)$ and $\tau_M(y, t)$ have been drawn against y for different values of t and the material constants λ, ν and fractional parameter α. For the sake of simplicity, all graphs are plotted by taking $U = 1, h = 1, \nu = 1, \mu = 1$. In Figs. 3 and 4, the diagrams of the velocity and the shear stress at three different times for fractional and ordinary Maxwell fluids are presented. As expected, both the velocity and the shear stress are increasing functions with respect to t and decreasing ones with respect to y for both type of fluids. The influence of the relaxation time λ on the fluid motion is shown in Figs. 5 and 6. Both the velocity and the shear stress are decreasing functions with respect to λ for fractional as well as ordinary Maxwell fluids. Figs. 7 and 8 show the variation of $u(y, t)$, $u_M(y, t)$, $\tau(y, t)$ and $\tau_M(y, t)$ with respect to kinematic viscosity ν. As it was to be expected, both the velocity and the shear stress are increasing functions with regard to ν. A strong influence of ν on the shear stress is clearly seen near

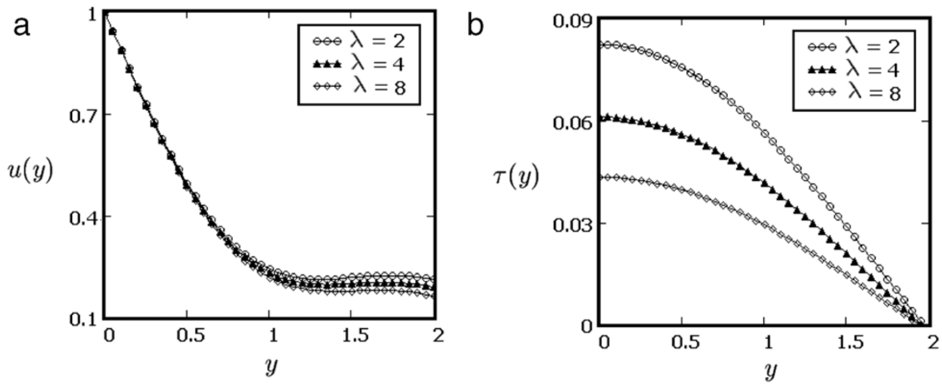

Fig. 5. Profiles of the velocity field $u(y, t)$ and the shear stress $\tau(y, t)$ for fractional Maxwell fluid given by Eqs. (23) and (31), for $\alpha = 0.4$, $t = 2$ s and different values of λ.

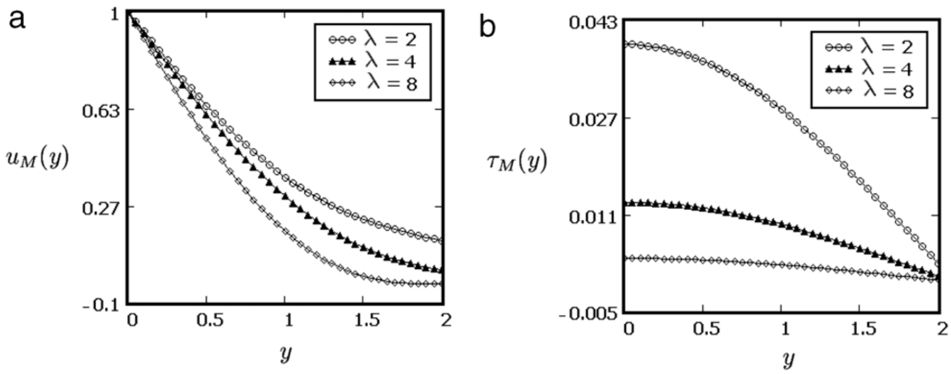

Fig. 6. Profiles of the velocity field $u_M(y, t)$ and the shear stress $\tau_M(y, t)$ for ordinary Maxwell fluid given by Eqs. (35) and (36), for $\alpha = 1$, $t = 2$ s and different values of λ.

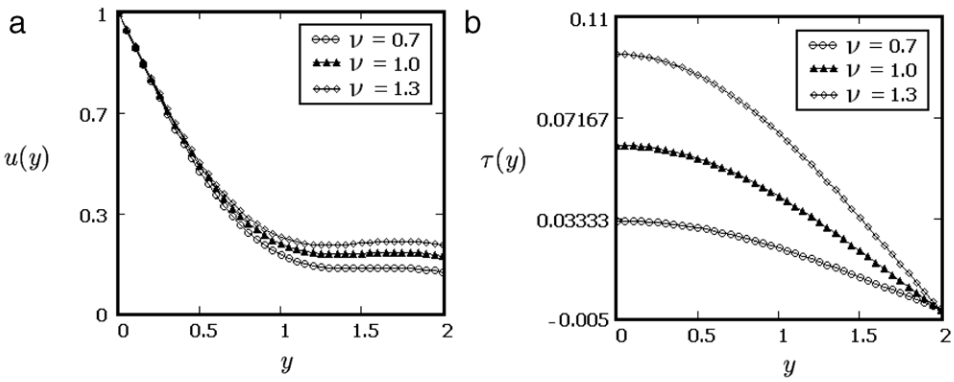

Fig. 7. Profiles of the velocity field $u(y, t)$ and the shear stress $\tau(y, t)$ for fractional Maxwell fluid given by Eqs. (23) and (31), for $\rho = 1$, $\lambda = 4$, $\alpha = 0.4$, $t = 2$ s and different values of ν.

the plate. Very important for us is to see the influence of the fractional parameter α on the fluid motion. The velocity, as expected, is an increasing function with respect to α, while the shear stress decreases with regard to α as shown in Fig. 9.

Finally, for comparison, the profiles of the velocity and the shear stress corresponding to the three models (fractional Maxwell, ordinary Maxwell, Newtonian) are together depicted in Fig. 10 for the same values of t and of the common material constants. It is clearly seen from these figures that, as expected the Newtonian fluid is the swiftest and the fractional Maxwell fluid is the slowest. Of course, these results are entirely agreed with those resulting from Figs. 5, 6 and 9. The choice of a reasonable value of the fractional parameter alpha, corresponding to the optimum dynamical model, results by comparison with the experimental data. The units of the material constants in all figures are SI units.

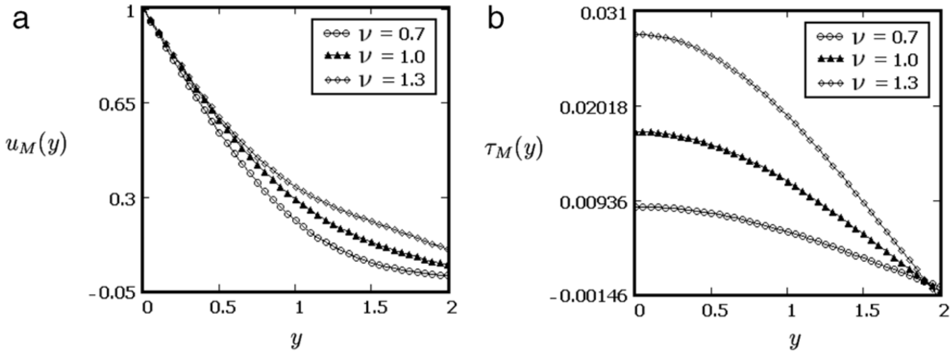

Fig. 8. Profiles of the velocity field $u_M(y, t)$ and the shear stress $\tau_M(y, t)$ for ordinary Maxwell fluid given by Eqs. (35) and (36), for $\rho = 1, \lambda = 4, \alpha = 1, t = 2$ s and different values of ν.

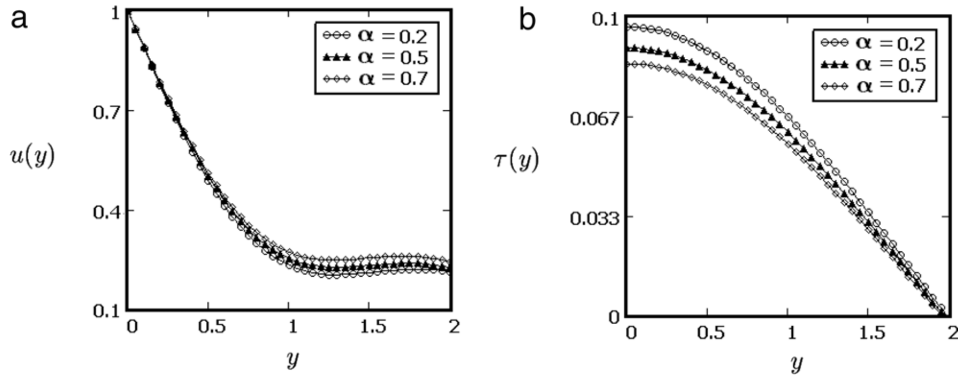

Fig. 9. Profiles of the velocity field $u(y, t)$ and the shear stress $\tau(y, t)$ for fractional Maxwell fluid given by Eqs. (23) and (31), for $\lambda = 1.5, t = 2$ s and different values of α.

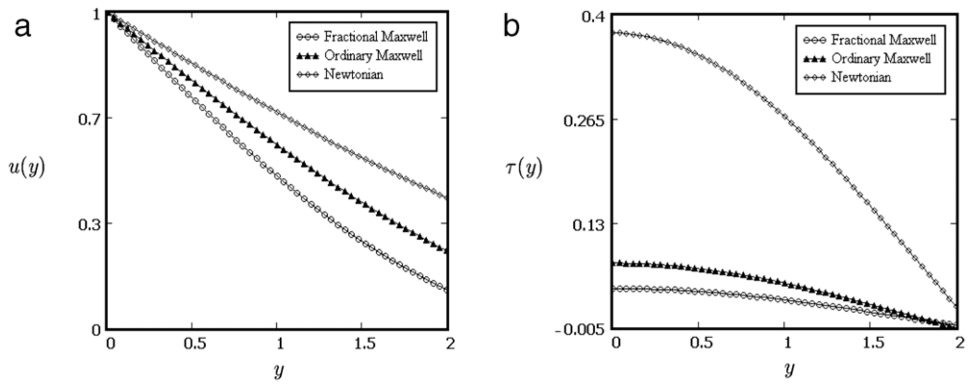

Fig. 10. Profiles of the velocity field $u(y, t)$ and the shear stress $\tau(y, t)$ for fractional Maxwell, ordinary Maxwell and Newtonian fluids, for $\lambda = 3, \alpha = 0.1$ and $t = 3$ s.

6. Concluding remarks

In this paper, the unsteady flow of a fractional Maxwell fluid over an infinite plate is studied by means of the Fourier sine and Laplace transforms. The motion of the fluid is due to the plate that at time $t = 0^+$ is suddenly moved with a constant velocity U in its plane. Exact analytical solutions are obtained for the velocity $u(y, t)$ and the shear stress $\tau(y, t)$ under integral and series form in terms of the generalized $G_{a,b,c}(\bullet, t)$ functions. These solutions, presented as a sum of the Newtonian solutions and the corresponding non-Newtonian contributions, satisfy all imposed initial and boundary conditions. For $\lambda \to 0$, as expected, they tend to the Newtonian solutions. The corresponding solutions for ordinary Maxwell fluids are also obtained from general solutions for $\alpha \to 1$. As a check of our calculi, we showed that our solution (35) is equivalent to those obtained in [35, Eq. (11)] and [37, Eq. (5.81)]. Indeed, as it is clearly seen from Figs. 1 and 2, their diagrams are identical. Finally, the influence of material and fractional parameters on the fluid motion are discussed through graphical illustrations. Special attention has been focused to analyze the difference between fractional and ordinary Maxwell fluids. The major finding of the present study are the following.

- The general solutions (23) and (31) are presented as a sum of the Newtonian solutions and the corresponding non-Newtonian contributions. These solutions can be easily particularized to give the similar solutions for ordinary Maxwell fluid. For $\lambda \to 0$, all solutions tend to the corresponding solutions for Newtonian fluids.
- The special solution (35) for the velocity field of ordinary Maxwell fluids, is equivalent to the known solution from the literature [30, Eq. (11)] and [37].
- Both velocities $u(y,t)$, $u_M(y,t)$ and the shear stresses $\tau(y,t)$, $\tau_M(y,t)$ as expected, are decreasing functions with respect to the relaxation time λ.
- The two entities, the velocity field and the shear stress for fractional and ordinary Maxwell fluids, are increasing functions with respect to the kinematic viscosity ν.
- The velocity $u_M(y,t)$ of the ordinary Maxwell fluids increases more rapidly in comparison with the velocity $u(y,t)$ of the fractional Maxwell fluids.
- The fractional parameter α has strong influence on the fluid motion. Generally the values of the fractional parameters α are not constant. However, the choice of a suitable value of this parameter, corresponding to the most favorable fractional dynamical model, results by comparison with the experimental data.
- A comparison between the three models clearly shows that the Newtonian fluid is the swifted and the fractional Maxwell fluid is the slowest.
- Generally, the presentation of the solutions as a sum between the Newtonian solutions and the corresponding non-Newtonian contributions is useful for those who want to bring to light the non-Newtonian effects on the fluid motion. They can easily verify if their effects diminish and disappear in time.

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